St. Xaviers College Jaipur

Department of Science
Mathematics

UG0803-MAT-51T-101
Semester-1
Discrete Mathematics and Optimization Techniques

Unit-1

1. If A and B are any two sets, then prove that :

   (i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

2. If a, b, c are any three arbitrary elements of the Boolean algebra $< B, +, ., ' >$, then prove that : $a + (b + c) = (a + b) + c$

3. State Pigeonhole principle. Assuming a patient is given a prescription of 45 tablets with the instructions to take atleast one tablet per day for 30 days. Prove that there must be a period of consecutive days during which the patient takes a total of exactly 14 tablets.

4. Show by means of truth table, that the compound statement $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology.

5. Prove that a relation R on a non empty set A is symmetric if and only if $R = R^{-1}$.

6. Prove that the dual of a complemented lattice is also a complemented lattice.

7. Prove that in a Boolean algebra $<B, +, \cdot, ', 0, 1>$, for each pair of elements $a, b \in B$.

   (i) $a + (a \cdot b) = a$ (ii) $a \cdot (a + b) = a$

8. Let R and S be two relations from the set A to the set B. Show tha

   (i) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$

   (ii) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

9. If $(B, \leq)$ is a distributive lattice, then prove that for all elements a, b, c $\in B$ $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \wedge a)$. 4,5½

10. If p, q are any two statements then show that $(\sim p \wedge (p \vee q)) \rightarrow q$ is a tautology.

11. Find the solution of the following recurrence relation :

$$a_r - 5a_{r-1} + 6a_{r-2} = 4^r.$$

12. Find the solution of the following recurrence relation :

$$a_r = 3a_{r-1} - 2a_{r-2}; \ r \geq 2 \quad a_1 = 5, \ a_2 = 3.$$

Unit-2

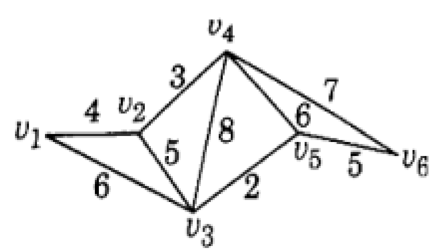
1. Determine the numeric function corresponding to the following generating function :

$$G(x) = \frac{1 + 2x}{2 + 3x + x^2}$$

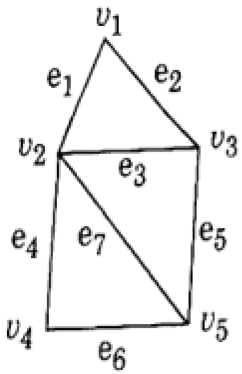
2. Solve the recurrence relation :
$a_r = 3a_{r-1} - 2a_{r-2}$; $r \geq 2$, $a_1 = 5$, $a_2 = 3$

3. If G is simple connected planar graph with n vertices and e edges (e>2), then prove that $e \leq 3n - 6$:

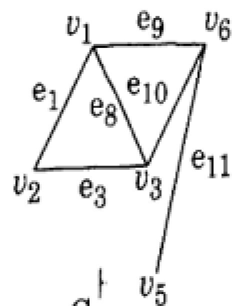
4. Find the shortest path and distance between the vertices V_1 and V_6 in the following weighted graph.



5. Find the union of graphs G_1 and G_2 shown in figure :



G_1



G_2

6. Define discrete numeric function. If a and b are two numeric function and if $c = a \cdot b$ and $d = a/b$, then prove that :

(i) $\Delta c_r = a_{r+1} \Delta b_r + b_r \Delta a_r$

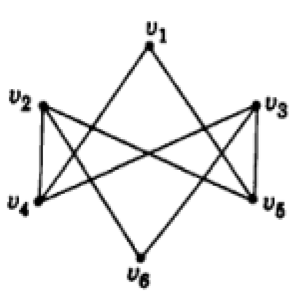
(ii) $\Delta d_r = \dfrac{b_r \Delta a_r - a_r \Delta b_r}{b_r . b_{r+1}}$

7. Determine the numeric function corresponding to the following generating function

$$G(x) = \frac{1 + x^3}{(1 - x)^4}.$$

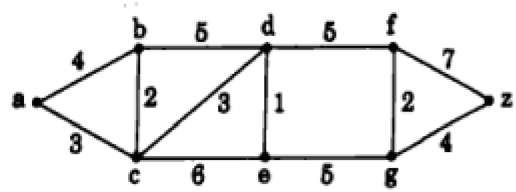
8. Prove that a simple graph G with n vertices and k components $(k \geq 1)$ can have at most $\frac{1}{2}(n-k)(n-k+1)$ edges.

9. Define complement of a simple graph G. Find the complement of the graph G shown in figure.



G

10. **Find the shortest path and distance between the vertices a and z in the following weighted graph.**



11. Show that:

$p \to (q \to r) \Leftrightarrow (p \wedge q) \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

12. Solve the following recurrence relation:

$a_{r+2} - 5a_{r+1} + 6a_r = 5r; r \geq 0$

13. Solve the following recurrence relation by the method of generating functions:

$a_{r+2} - 3a_{r+1} + 2a_r = 0; r \geq 0, a_0 = 2, a_1 = 3$

14. Let G be an undirected graph with 12 edges. If G has 6 vertices each of degree 3 and the rest vertices have degree less than 3, then what is the minimum number of vertices G can have? 4,5

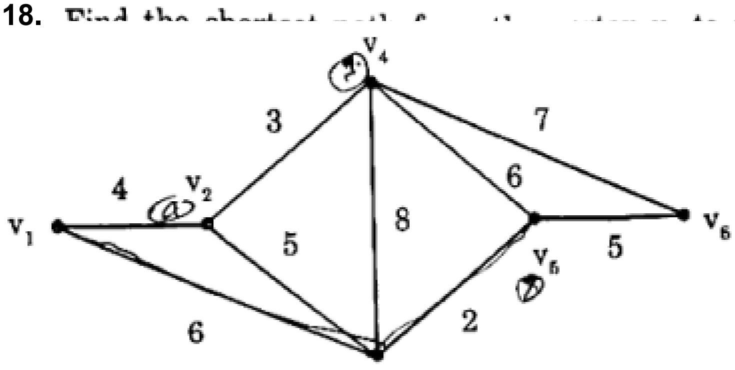
15. Prove that a connected graph G contains an Eulerian path if and only if it has exactly two vertices of odd degree.

16. Prove that there are even number of vertices of odd degree in any indirected graph.

17. If G is a simple connected planar graph with n vertices ad e edges then prove that :

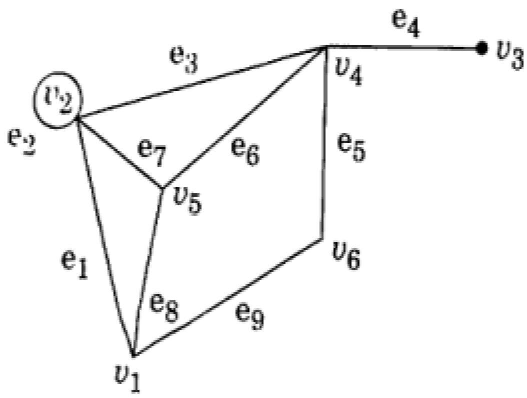
$$e \leq 3n - 6 \qquad (e \geq 2).$$

18. Find the shortest [illegible] x v_6 in the following weighted



Unit-3

19. Find the adjacency matrix of the following graph :



2. Define Tree. prove that a tree with n vertices has exactly (n – 1) edges.

3. Prove that every tree has either one or two centres.

4. Determine whether the graphs given below are Euler, Hamiltonian or not.

Fig.1

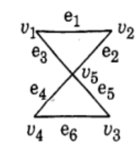
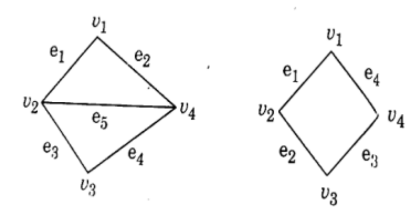
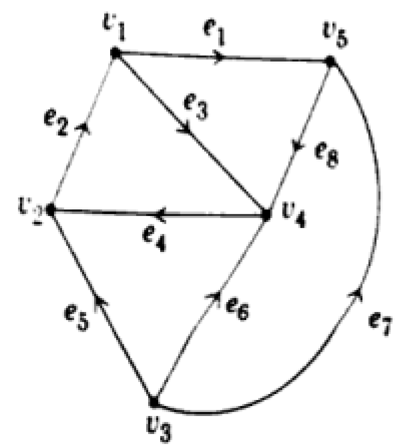


Fig. 2



5. Find the adjacency matrix and the incidence matrix of the following directed graph



6.

If h is the height of a balanced complete binary tree of n vertices then prove that

$$h = \log_2\left(\frac{n+1}{2}\right).$$

7. Let G be a connected graph on n vertices and e edges. Prove that G has a Hamiltonian path if.

$$e \geq \frac{1}{2}\left(n^2 - 3n + 6\right)$$

8. A tree T has n_1 vertices of degree 1. n_2 vertices of degree 2. n_3 vertices of degree 3, ..., n_k vertices of degree k. Prove that $n_1 = 2 + n_3 + 2n_4 + 3n_5 + \ldots\ldots\ldots + (k-2)\,n_k$.

9. Define incidence matrix of a digraph. Draw the digraph, incidence matrix is shown below:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9
u_1	−1	−1	0	1	1	0	0	0	0
u_2	1	0	1	0	0	0	−1	0	0
u_3	0	1	−1	0	0	−1	0	0	1
u_4	0	0	0	−1	0	1	1	1	0
u_5	0	0	0	0	−1	0	0	−1	−1

10.

Define with example :

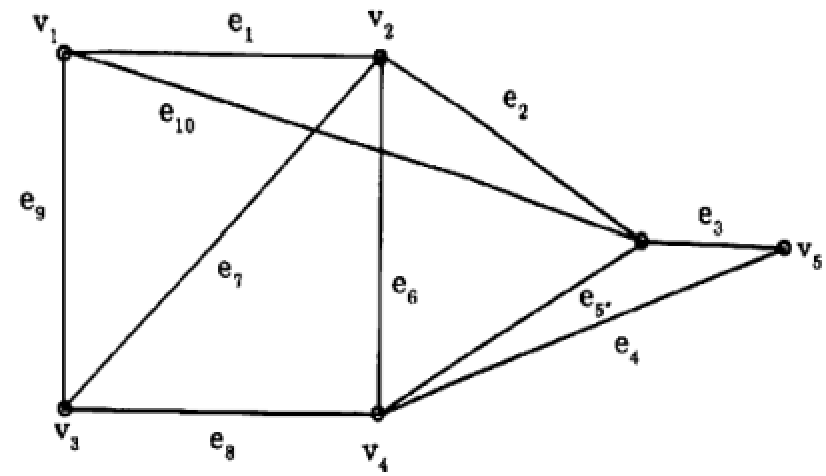
(i) Union of two graph

(ii) Product of two graph

(iii) Join of two graph

(iv) Plannar graph.

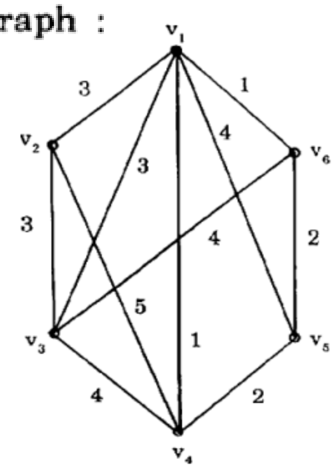
11. Find the adjacency matrix of the following simple graph :



12. If T is binary tree with n vertices and of height h then prove that :

$$h + 1 \leq n \leq 2^{h+1} - 1.$$

13. Find the minimal spanning tree for the following graph :



Unit -4

Solve by Two-Phase Method:

a) Minimise $12{:}5x_1 + 14.5x_2$.

Subject to :

$$x_1 + x_2 \geqslant 2000$$
$$0.4x_1 + 0.75x_2 \geqslant 1000$$
$$0.075x_1 + 0.1x_2 \leqslant 200$$
$$x_1 \geqslant 0, \; x_2 \geqslant 0$$

b) Maximise $12x_1 + 15x_2 + 9x_3$

Subject to :

$$8x_1 + 16x_2 + 12x_3 \leqslant 250$$
$$4x_1 + 8x_2 + 10x_3 \geqslant 80$$
$$7x_1 + 9x_2 + 8x_3 = 105$$
$$x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0.$$

c) Minimize $z = -3x_1 + x_2 - 2x_3$

subject to

$$x_1 + 3x_2 + x_3 \leq 5$$
$$2x_1 - x_2 + x_3 \geq 2$$
$$4x_1 + 3x_2 - 2x_3 = 5$$

$$x_1, x_2, x_3 \geq 0$$

d) Minimize $Z = x_1 - 2x_2 - 3x_3$
subject to
$-2x_1 + x_2 + 3x_3 = 2$,
$2x_1 + 3x_2 + 4x_3 = 1$
and $x_1 > 0$, $x_2 > 0$, $x_3 > 0$.

e) Maximize $Z = 2x_1 - x_2 + x_3$

subject to the constraints :
$$x_1 + x_2 - 3x_3 \leq 8,$$
$$4x_1 - x_2 + x_3 \geq 2,$$
$$2x_1 + 3x_2 - x_3 \geq 4$$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

2. Solve by Simplex Method

a) Maximise $2000x_1 + 3000x_2$

Subject to :
$$6x_1 + 9x_2 \leqslant 100$$
$$2x_1 + x_2 \leqslant 20$$
$$x_1 \geqslant 0,\ x_2 \geqslant 0$$

b) Maximise $5x_1 + 4x_2$

Subject to :
$$x_1 \leqslant 7$$
$$x_1 - x_2 \leqslant 8$$
$$x_1 \geqslant 0,\ x_2 \geqslant 0.$$

c) Maximise $50x_1 + 60x_2$

Subject to :

$$2x_1 + x_2 \leqslant 300$$
$$3x_1 + 4x_2 \leqslant 509$$
$$4x_1 + 7x_2 \leqslant 812$$
$$x_1 \geqslant 0, \quad x_2 \geqslant 0$$

d) Maximise $12x_1 + 3x_2 + x3$

Subject to :

$$10x_1 + 2x_2 + x_3 \leqslant 100$$
$$7x_1 + 3x_2 + 2x_3 \leqslant 77$$
$$2x_1 + 4x_2 + x_3 \leqslant 80$$
$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0.$$

e)

$$\text{Maximize} \quad P = 5x_1 + 4x_2$$
$$\text{subject to} \quad 4x_1 + 2x_2 \leq 32$$
$$2x_1 + 3x_2 \leq 24$$
$$x_1, x_2 \geq 0$$

f)

$$\text{Maximize} \quad P = 6x_1 + 3x_2$$
$$\text{subject to} \quad -2x_1 + 3x_2 \leq 9$$
$$-x_1 + 3x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

Solve the following by converting the primal to dual:

a) Maximise $12x_1 + 3x_2 + x_3$

Subject to .

$$10x_1 + 2x_2 + x_3 \leqslant 100$$
$$7x_1 + 3x_2 + 2x_3 \leqslant 77$$
$$2x_1 + 4x_2 + x_3 \leqslant 80$$
$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0.$$

b) Maximise $12x_1 + 15x_2 + 9x_3$

Subject to :

$$8x_1 + 16x_2 + 12x_3 \leqslant 250$$
$$4x_1 + 8x_2 + 10x_3 \geqslant 80$$
$$7x_1 + 9x_2 + 8x_3 = 105$$
$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0.$$

c) Maximize $z = 3x_1 + 2x_2$

subject to

$-x_1 + 2x_2 \leq 4$
$3x_1 + 2x_2 \leq 14$
$x_1 - x_2 \leq 3$

$x_1, x_2 \geq 0$

d)
$$\begin{aligned} \text{Maximize} \quad & P = 4x_1 + 3x_2 + 2x_3 \\ \text{subject to} \quad & 3x_1 + 2x_2 + 5x_3 \leq 23 \\ & 2x_1 + x_2 + x_3 \leq 8 \\ & x_1 + x_2 + 2x_3 \leq 7 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

e) Maximize $Z = f(x,y) = 3x + 2y$
subject to: $2x + y \leq 18$
$2x + 3y \leq 42$
$3x + y \leq 24$
$x \geq 0 , y \geq 0$

Solve the Assignment Problem

a)

		Programmes			
		A	**B**	**C**	**D**
Programmers	**1**	120	100	80	90
	2	80	90	110	70
	3	110	140	120	100
	4	90	90	80	90

b)

Deficit City / Surplus city	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

c)

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Solve by transportation problem

a)

		Destination				
		D1	D2	D3	D4	Supply
Source	O1	3	1	7	4	300
	O2	2	6	5	9	400
	O3	8	3	3	2	500
Demand:		250	350	400	200	1200